ON A GENERALISATION OF THE NON-LINEAR DIFFERENTIAL EQUATION

$$\frac{d^2u}{dt^2} - \varepsilon(1-u^2)\frac{du}{dt} + u = 0$$

BY

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§ 1. In 1920 [1]) and later we studied the differential equation

$$\frac{d^2u}{dt^2} - \varepsilon\psi(u)\frac{du}{dt} + u = 0, \tag{1}$$

and in particular

$$\frac{d^2u}{dt^2} - \varepsilon(1-u^2)\frac{du}{dt} + u = 0. \tag{2}$$

It was shown that, for $\varepsilon \ll 1$, the latter equation has as approximate solution a sinusoïdal oscillation which builds up slowly towards a final amplitude of numerical value 2 according to

$$u = \frac{2\cos t}{\sqrt{1+e^{-\varepsilon(t-t_0)}}}. \tag{3}$$

For large ε however ($\varepsilon \gg 1$) it yielded as solution a rather discontinuous type of motion which were baptized "relaxation oscillations" and which were characterized by the fact that they reach their final periodic form in a few oscillations only. It was further shown that with the latter condition ($\varepsilon \gg 1$), contrary to what is the case for $\varepsilon \ll 1$, external periodic forces could easily influence the frequency of the system (frequency demultiplication), whereas its amplitude was rather stable.

Applications to biological rhythms were investigated, [2]) especially to the beating of the human heart, where some known abnormal rhythms could easily be explained both theoretically and experimentaly and some new rhythms could be predicted.

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[1]) BALTH. VAN DER POL, „Over de amplitude van vrije en gedwongen triode-trillingen", Tijdschr. Ned. Rad. Gen., 1, 3 (1920). English version in: BALTH. VAN DER POL, "A theory of the amplitude of free and forced triode vibrations", Radio Review, 1, 701, 754 (1920).

[2]) BALTH. VAN DER POL and J. VAN DER MARK, "The heartbeat considered as a relaxation oscillation and an electrical model of the heart", Phil. Mag., 6, 763 (1928).

In the interim extensive researches on the properties of the solutions of (1) and (2) have appeared in the litterature. In connection herewith we refer to the forthcoming book by N. MINORSKY.

It is the purpose of the present note a) to generalize (2) to a non-linear *partial* differential equation and b) to develop from it solutions which show a remarkable analogy with the propagation of electrical impulses on nerve fibres.

The generalisation is the following

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \varepsilon(1-u^2)\frac{\partial u}{\partial t}} \qquad (\varepsilon > 0). \tag{4}$$

Here, as before, t represents the time, x however is a coordinate measured along a nerve fibre.

This equation (4) shows an analogy with the one-dimensional wave-equation, but, instead of representing a positive damping, it contains a negative damping corresponding to the energy liberated in the nervous system everywhere locally.

In (4) the undamped ($\varepsilon = 0$) propagation velocity is taken to be unity. It can of course be adjusted to any value provided a proper unit of length is introduced.

§ 2. In order to simplify our further analysis somewhat we put $u(x, t) = \sqrt{3}\, y(x, t)$, so that then (4) reads

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} - \varepsilon \frac{\partial}{\partial t}(y - y^3). \tag{5}$$

In order to study disturbances propagating in one direction only we try a solution of (5) of the following form

$$y = \psi(x - vt) = \psi(s), \tag{6}$$

where

$$s = x - vt,$$

and v is the velocity of propagation.
With these variables we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2\psi}{ds^2},$$

$$\frac{\partial y}{\partial t} = -v\frac{d\psi}{ds},$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{d^2\psi}{ds^2},$$

so that (5) becomes

$$\frac{d^2\psi}{ds^2} = v^2\frac{d^2\psi}{ds^2} + \varepsilon v\frac{d}{ds}(\psi - \psi^3),$$

or

$$(v^2-1)\frac{d^2\psi}{ds^2}+\varepsilon v\frac{d}{ds}(\psi-\psi^3)=0,$$

or

$$\frac{d^2\psi}{ds^2}-\varepsilon'\frac{d}{ds}(\psi-\psi^3)=0, \tag{7}$$

where we introduced the "effective" ε' as

$$\varepsilon'=\frac{\varepsilon v}{1-v^2}. \tag{7a}$$

The equation (7) can be integrated completely. For, after a first integration it leads to

$$\frac{d\psi}{ds}-\varepsilon'(\psi-\psi^3)=K, \tag{8}$$

where K is a constant.

Consider the case $K=0$. Then (8) can be written after multiplication by 2ψ as

$$\frac{d\psi^2}{ds}-2\varepsilon'(\psi^2-\psi^4)=0. \tag{9}$$

In order to solve (9) we substitute $\psi^2=\frac{1}{z}$. This yields

$$\frac{dz}{ds}+2\varepsilon'(z-1)=0,$$

yielding

$$z=1+e^{-2\varepsilon'(s-s_0)},$$

where s_0 is an immaterial integration constant.

Hence we obtain

$$\psi=\pm\frac{1}{\sqrt{1+e^{-2\varepsilon'(s-s_0)}}}. \tag{10}$$

Returning to the original variables (10) gives as a solution of (4)

$$u(x,t)=\frac{\pm\sqrt{3}}{\sqrt{1+e^{-2\varepsilon\frac{v}{1-v^2}(x-vt)}}}, \tag{11}$$

where we have put $s_0=0$, which represents nothing but a shift in the origin of time.

Clearly (11) represents a disturbance propagating along the x coordinate with a velocity v.

It is of interest to remark that the "effective" $\varepsilon'=\frac{\varepsilon v}{1-v^2}$ is the larger the nearer v is to unity. Under these circumstances and for fixed x, (11) shows that $u(x,t)$ presents a *very sudden* drop from $\pm\sqrt{3}$ to zero,

when vt is in the neighbourhood of x. The time (or space) derivative of $u(x, t)$ which we may associate with the electrical disturbances as being propagated over the nerves, therefore shows a sharp impulse being propagated with the velocity v. The impulse will be sharper the nearer v^2 is equal to unity, independent of whether the original ε is large or small. It seems therefore appropriate to associate $\partial u/\partial x$ or $\partial u/\partial t$ with the electrical potentials found in the nervous system, in which case a striking analogy with the biological phenomena is represented by the non-linear partial differential equation (4).